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DETACHMENT OF A BEAM GLUED TO A RIGID PLATE

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The contact problem of detachment of an elastic beam glued to a rigid plate is considered. A mathematical model is proposed and theoretical results are compared with experimental data.

Key words: *elastic beam, detachment, contact problem, plate, Legendre polynomials.*

Introduction. The present paper addresses a contact problem of detachment of an elastic beam glued to a rigid plate.

Various methods for solving contact problems of elastic plates and shells have been proposed (see, e.g., [1–3]). Below, we use the equations of elastic deformation of plates and shells [4, 5] obtained by expanding unknown functions into series in terms of Legendre polynomials. A specific feature of this approach is that several approximations are used for the same unknown functions. Arbitrary conditions for stresses, displacements, and mixed conditions can be specified on the front surfaces without reducing the differential order of equations. This allows one to adequately formulate the matching conditions at the boundary of exfoliation and gluing zones. These equations were used to solve, for example, a plane contact problem for an elastic layer [6].

Equations of Deformation of an Elastic Beam. We consider the case of plane stresses. The stresses are approximated by truncated series in Legendre polynomials $P_k(\xi)$ ($\xi = y/h$):

$$
\begin{gathered}
2h\sigma_x = N + (3M/h)P_1(\xi), \qquad \sigma_y = p_0 + \Delta p P_1(\xi),\\
2h\sigma_{xy} = Q + 2h\Delta q P_1(\xi) + (2hq_0 - Q)P_2(\xi),\\
\Delta p = 0.5(p^+ - p^-), \qquad p_0 = 0.5(p^+ + p^-),\\
\Delta q = 0.5(q^+ - q^-), \qquad q_0 = 0.5(q^+ + q^-).
\end{gathered}
\tag{1}
$$

Here $N = \displaystyle\int_{-h}^{h} \sigma_x\,dy$ is the force, $M = \displaystyle\int_{-h}^{h} \sigma_x y\,dy$ is the moment, $Q = \displaystyle\int_{-h}^{h} \sigma_{xy}\,dy$ is the transverse shear force, and $p^\pm$ and $q^\pm$ are the normal and shear stresses in the contact planes ($\xi = \pm 1$).

Displacements and strains are approximated by the truncated series

$$
\begin{gathered}
u_x = u + \psi P_1(\xi) + (u_0 - u)P_2(\xi) + (\Delta u - \psi)P_3(\xi), \qquad u_y = v + \Delta v P_1(\xi) + (v_0 - v)P_2(\xi),\\
e_x = \frac{du}{dx} + \frac{d\psi}{dx}\,P_1(\xi), \qquad e_y = \frac{1}{h}\,\Delta v + \frac{3}{h}\,(v_0 - v)P_1(\xi),\\
e_{xy} = \frac{dv}{dx} + \frac{1}{h}\,\Delta u + \frac{3}{h}\,(u_0 - u)P_1(\xi) + \frac{5}{h}\,(\Delta u - \psi)P_2(\xi),\\
\Delta u = 0.5(u^+ - u^-), \qquad u_0 = 0.5(u^+ + u^-),\\
\Delta v = 0.5(v^+ - v^-), \qquad v_0 = 0.5(v^+ + v^-).
\end{gathered}
\tag{2}
$$

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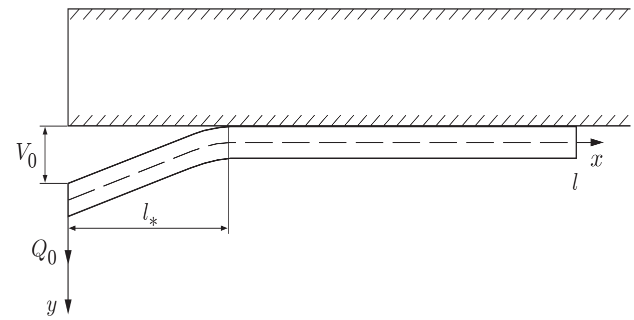

Fig. 1

Here $u = \frac{1}{2}\int\limits_{-1}^{1} u_x\,d\xi$ and $v = \frac{1}{2}\int\limits_{-1}^{1} u_y\,d\xi$ are the displacements averaged over the thickness, $\psi = \frac{3}{2}\int\limits_{-1}^{1} u_x\xi\,d\xi$ is the angle of rotation of the normal to the mid-plane $y = 0$, and $v^{\pm}$ and $u^{\pm}$ are the normal and tangential displacements at the contact surfaces ($\xi = \pm 1$), respectively.

The unknown functions that enter the coefficients in the polynomials in formulas (1) and (2) are determined from the system of equations [7] that comprises:

— the equations of equilibrium

$$\frac{dN}{dx} + 2\Delta q = 0, \qquad \frac{dM}{dx} - Q + 2hq_0 = 0, \qquad \frac{dQ}{dx} + 2\Delta p = 0; \tag{3}$$

— the differential equations derived from Hooke's law:

$$\frac{du}{dx} = \frac{N}{2hE} - \nu\,\frac{p_0}{E}, \quad \frac{d\psi}{dx} = \frac{3M}{2h^2E} - \nu\,\frac{\Delta p}{E}, \quad \frac{dv}{dx} + \frac{\Delta u}{h} = \frac{Q}{2h\mu}; \tag{4}$$

— the algebraic equations derived from Hooke's law:

$$\begin{gathered} u_0 - u = \frac{h}{3\mu}\,\Delta q, \qquad \Delta u - \psi = \frac{h}{5\mu}\Big(q_0 - \frac{Q}{2h}\Big), \\ \Delta v = h\,\frac{p_0}{E} - \nu\,\frac{N}{2E}, \qquad v_0 - v = h\,\frac{\Delta p}{3E} - \nu\,\frac{M}{2hE}. \end{gathered} \tag{5}$$

Here E and μ are Young's and shear moduli, respectively, and ν is Poisson's ratio.

The system of ordinary differential equations (3) and (4) for the unknown functions N, M, Q, u, ψ, and v has the sixth order.

Detachment of a Beam from a Rigid Plate. A beam of unit width, thickness $2h$, and length l is glued to a rigid plate. A force Q_0 is applied to the edge $x = 0$. An increase in Q_0 leads to detachment of the beam from the plate. There are two zones: exfoliation zone ($0 \leqslant x \leqslant l_*$) and zone where the beam remains glued to the plate ($l_* < x \leqslant l$) (Fig. 1).

The following boundary conditions are specified at the ends of the beam:

$$N = 0, \qquad M = 0, \qquad Q = -Q_0 \qquad \text{for} \quad x = 0; \tag{6}$$

$$N = 0, \qquad M = 0, \qquad Q = 0 \qquad \text{for} \quad x = l. \tag{7}$$

The front surface $\xi = 1$ is stress-free:

$$q^+ = 0, \qquad p^+ = 0 \qquad \text{for} \quad 0 \leqslant x \leqslant l. \tag{8}$$

At the front surface $\xi = -1$, the following conditions are formulated. In the exfoliation zone, we have

$$q^- = 0, \qquad p^- = 0 \qquad \text{for} \quad 0 \leqslant x \leqslant l_* \tag{9}$$

and in the zone where the beam remains glued to the plate (for brevity, we call it the gluing zone):

$$u^- = 0, \qquad v^- = 0 \qquad \text{for} \quad l_* < x \leqslant l. \tag{10}$$

Moreover, we assume that the normal stresses satisfy the inequality $p^- < \sigma_0$ at each point in the gluing zone (σ_0 is a certain positive quantity that characterizes the strength properties of the gluing layer). If the normal stresses at a certain point reaches σ_0, exfoliation occurs at this point. Thus, we obtain the equality

$$p^- = \sigma_0 \qquad \text{for} \quad x = l_*. \tag{11}$$

The boundary-value problem for system (1)–(5) subject to the boundary conditions (6)–(10) is difficult to solve. To simplify the formulation of the initial problem, we make the following assumptions. The unknown functions u and N do not play the decisive role and can be ignored. Moreover, we set $\nu\Delta p/E = \nu M/(2hE) = 0$ in Eqs. (4) and (5).

With these assumptions in mind, we reduce system (1)–(5) to the system

$$\frac{dM}{dx} - Q + 2hq_0 = 0, \qquad \frac{dQ}{dx} + 2\Delta p = 0; \tag{12}$$

$$\frac{d\psi}{dx} = \frac{3M}{2h^2E}, \qquad \frac{dv}{dx} + \frac{\Delta u}{h} = \frac{Q}{2h\mu}; \tag{13}$$

$$u_0 = \frac{h}{3\mu}\,\Delta q, \qquad \Delta u - \psi = \frac{h}{5\mu}\Big(q_0 - \frac{Q}{2h}\Big),$$
$$\Delta v = h\,\frac{p_0}{E}, \qquad v_0 - v = h\frac{\Delta p}{3E}. \tag{14}$$

We consider the solution of the problem in the exfoliation zone ($0 \leqslant x \leqslant l_*$). Equations (12)–(14) are supplemented by the boundary conditions (8) and (9). The solution of the resulting system with the boundary conditions (6) has the form

$$Q = -Q_0, \qquad M = -xQ_0, \qquad \psi = -\frac{3}{4h^2}\,\frac{Q_0}{E}\,x^2 + \Psi,$$
$$v = \frac{1}{4h^3}\,\frac{Q_0}{E}\,x^3 - \frac{6(1+\nu)}{5h}\,\frac{Q_0}{E}\,x - \frac{x}{h}\,\Psi + V. \tag{15}$$

Here Ψ and V are unknown constants.

In the gluing zone ($l_* < x \leqslant l$), Eqs. (12)–(14) should be supplemented by the boundary conditions (8) and (10), which can be written as

$$q_0 + \Delta q = 0, \quad p_0 + \Delta p = 0, \quad u_0 - \Delta u = 0, \quad v_0 - \Delta v = 0. \tag{16}$$

The set of equations (14) and (16) can be considered as a linear system of algebraic equations for the unknown functions Δu, u_0, Δv, v_0, Δp, p_0, Δq, and q_0, whose solution has the form

$$q_0 = -\Delta q = -\frac{15\mu}{8h}\,\psi + \frac{3}{16h}\,Q, \qquad p_0 = -\Delta p = \frac{3E}{4h}\,v,$$
$$u_0 = \Delta u = \frac{5}{8}\,\psi - \frac{1}{16h}\,Q, \qquad v_0 = \Delta v = \frac{3}{4}\,v. \tag{17}$$

Substituting (17) into Eqs. (12) and (13), we obtain the fourth-order system of ordinary differential equations

$$\frac{dM}{dx} = h\,\frac{15\mu}{4h}\,\psi + \frac{5}{8}\,Q, \qquad \frac{dQ}{dx} = \frac{3E}{2h}\,v,$$
$$\frac{d\psi}{dx} = \frac{3M}{2h^2E}, \qquad \frac{dv}{dx} = -\frac{5}{8h}\,\psi + \frac{9}{16h\mu}\,Q. \tag{18}$$

We denote the normal contact stress in the gluing zone by p. From (17), we obtain the normal contact stresses

$$p = 3Ev/(4h). \tag{19}$$

The characteristic equation of system (18) is a biquadratic equation that has four complex roots. The solution involves four constants. Let $l/h \gg 1$. In this case, we assume that the boundary conditions (7) are specified at infinity. After these conditions are satisfied, the solution of system (18) becomes

$$\psi = (C_1\cos(\beta x/h) + C_2\sin(\beta x/h))\exp(-\alpha x/h). \tag{20}$$

Here C_1 and C_2 are unknown constants and α and β are determined by the formulas

$$\alpha = 0.5\sqrt{c+d}, \qquad \beta = 0.5\sqrt{-c+d},$$

$$c = \frac{9}{16}\Big(\frac{10}{k} + \frac{3k}{2}\Big), \qquad d = \frac{3}{2}\sqrt{10}, \qquad k = \frac{E}{\mu} = 2(1+\nu).$$

The unknown functions M, Q, and v have the form similar to expression (20).

Solution (15) contains two unknown constants Ψ and V for the exfoliation zone and two unknown constants C_1 and C_2 for the gluing zone. These unknown constants are determined from the matching conditions for solutions (15) and (12) at the point $x = l_*$, which can be written as

$$[M] = 0, \quad [Q] = 0, \quad [\psi] = 0, \quad [v] = 0 \qquad \text{for} \quad x = l_*. \tag{21}$$

Here $[\,\cdot\,]$ denotes a discontinuity of the function.

The unknown boundary of the exfoliation zone l_* is determined from conditions (11) and (19). The constants C_1 and C_2 are calculated from the first two equalities of (21). Once these are found, the constants Ψ and V are determined from the remaining relations. Equalities (11) and (19) yield an expression for l_*.

Without derivation, we write the expressions

$$Q_0 = 2h\sigma_0 A/(5\eta/8 + B); \tag{22}$$

$$A = \frac{15}{8k} + \frac{\sqrt{10}}{4}, \qquad B = \frac{\sqrt{10}}{6}\sqrt{\frac{9}{8k}\Big(5 + \frac{3}{4}k^2\Big) + \frac{3}{2}\sqrt{10}};$$

$$V_0 = (Q_0/E)(4\eta^3 + a_2\eta^2 + a_1\eta + a_0); \tag{23}$$

$$a_0 = \frac{B}{3A}, \qquad a_1 = \frac{5}{18k}, \qquad a_2 = \frac{5}{8B}\Big(\frac{9k}{5} + \frac{16}{\sqrt{10}} + \frac{5}{24A}\Big),$$

where $\eta = l_*/(2h)$ is a dimensionless parameter determining the boundary of the exfoliation zone.

Formula (22) determines the relation between the applied load Q_0 and the size of the exfoliation zone l_*, and formula (23) determines the relation between the deflection of the beam end V_0 and the size of the exfoliation zone l_*.

Experiments. To verify the above-proposed mathematical model of detachment of an elastic beam from a rigid foundation, we performed experiments.

Specimens made of Plexiglas, glass-reinforced plastic, and Duralumin 3 mm thick, 10 mm wide, and 300 mm long were glued to a rigid plate with epoxy resin. The plate was made of transparent Plexiglas, which allowed us to determine the length of the exfoliation zone of the specimen with reasonable accuracy. Experimental conditions (specimen size, roughness of the surfaces to be glued, pressure, and gluing time) were identical. For each material, the experiment was performed three or more times.

Figure 1 shows schematically the experimental diagram. According to the loading conditions and characteristics measured, the experiments are divided into two types. In the first case, the force Q_0 was applied to the specimen end ("soft" loading). The force Q_0 was measured by a dynamometer and the corresponding length of the exfoliation zone l_* was measured visually (the plate was made of transparent Plexiglas). The boundary of the exfoliation zone was distinctly seen. The transparent epoxy-resin gluing layer became dimmed and was covered by fine cracks. In the second case, the end of the specimen was given a deflection V_0 ("stiff" loading). The deflection V_0 was measured with the help of a clock-type indicator and the value of l_* was determined visually.

At the initial stage of "soft" loading, the following is observed. The load increases to a certain critical value Q_*, and exfoliation does not occur ($l_* = 0$). At $Q_0 = Q_*$, the load decreases abruptly and the length of the exfoliation zone increases rapidly. Because of the process transiency, we failed to record the dependence between Q_0 and l_*. To determine this dependence, we used the following procedure.

We chose a sequence of lengths $0 < l_i < l_*$ for which the specimen was not glued in the interval from 0 to l_i, i.e., the exfoliation zone was modeled artificially. Then the specimen was loaded and the critical load Q_{i*} that made exfoliation continue was determined. Setting $Q_0 = Q_{i*}$ and $l_* = l_i$, we obtain a dependence between Q_0 and l_*. In the experiments, the values of l_i were equal to 20, 30, and 50 mm.

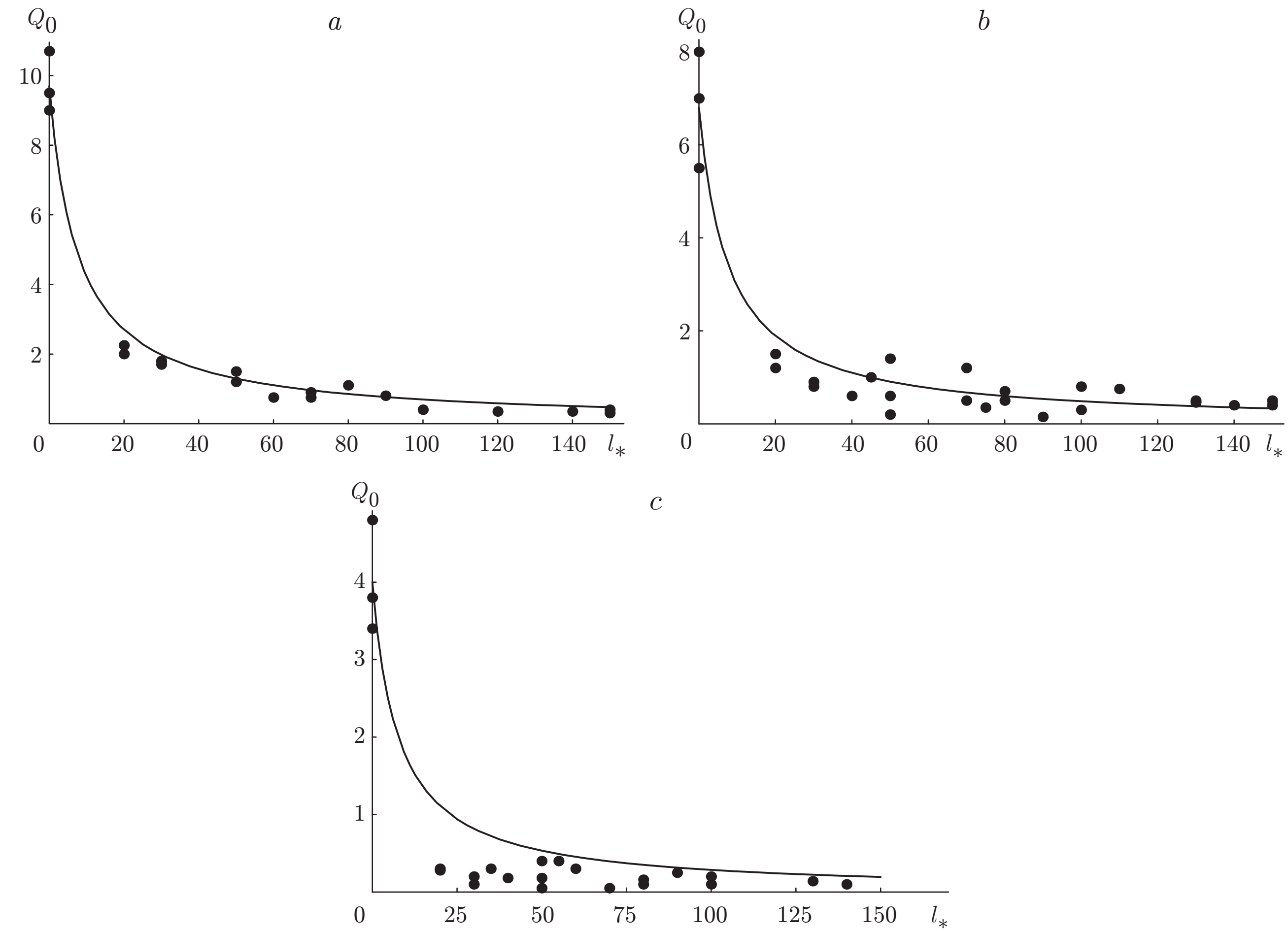

Fig. 2

For the "stiff" loading, a monotonic dependence between l_* and specified deflection at the beam end V_0 was observed for each specimen. The dimensions of the exfoliation zone increased with increasing deflection.

It should be noted that, for specimens made of the same material, the scatter of experimental values of the exfoliation-zone length becomes more pronounced with an increase in the deflection V_0.

Comparison of Theoretical and Experimental Results. To verify whether the mathematical model proposed above describes adequately the detachment of a beam glued to a rigid plate, we compare theoretical and experimental results.

We consider the case of "soft" loading. Taking into account the specimen width b, we write formula (22) as

$$Q_0 = \frac{Q_0'}{1 + \gamma l_*}, \qquad Q_0' = 2hb\sigma_0 \frac{A}{B}, \qquad \gamma = \frac{5}{16hB}. \tag{24}$$

Formula (24) contains a physical constant σ_0, which characterizes the strength properties of the gluing layer. This quantity depends on the shape and roughness of contacting surfaces, properties of the glue, characteristics of the bodies in contact, etc. It is, therefore, expedient to determine this quantity directly from experimental data. For $l_* = 0$, we have $Q_0 = Q_*$ (Q_* is the critical load). At the same time, it follows from formula (24) that $Q_0 = Q_0'$ for $l_* = 0$. Setting $Q_0' = Q_*$, we obtain

$$\sigma_0 = BQ_*/(2hbA). \tag{25}$$

Since the experimental values of Q_* differ for different specimens (scatter is smaller than 12%), the averaged value is used in the calculations and formula (25). We obtain $Q_* = 9.7$ kg for aluminum, $Q_* = 6.7$ kg for glass-fiber plastic, and $Q_* = 4$ kg for Plexiglas. In the calculations, Young's modulus E and Poisson's ratio ν were $E = 7 \cdot 10^4$ MPa and $\nu = 0.27$ for aluminum, $E = 2.8 \cdot 10^4$ MPa and $\nu = 0.25$ for glass-fiber plastic, and $E = 0.3 \cdot 10^4$ MPa and $\nu = 0.35$ for Plexiglas.

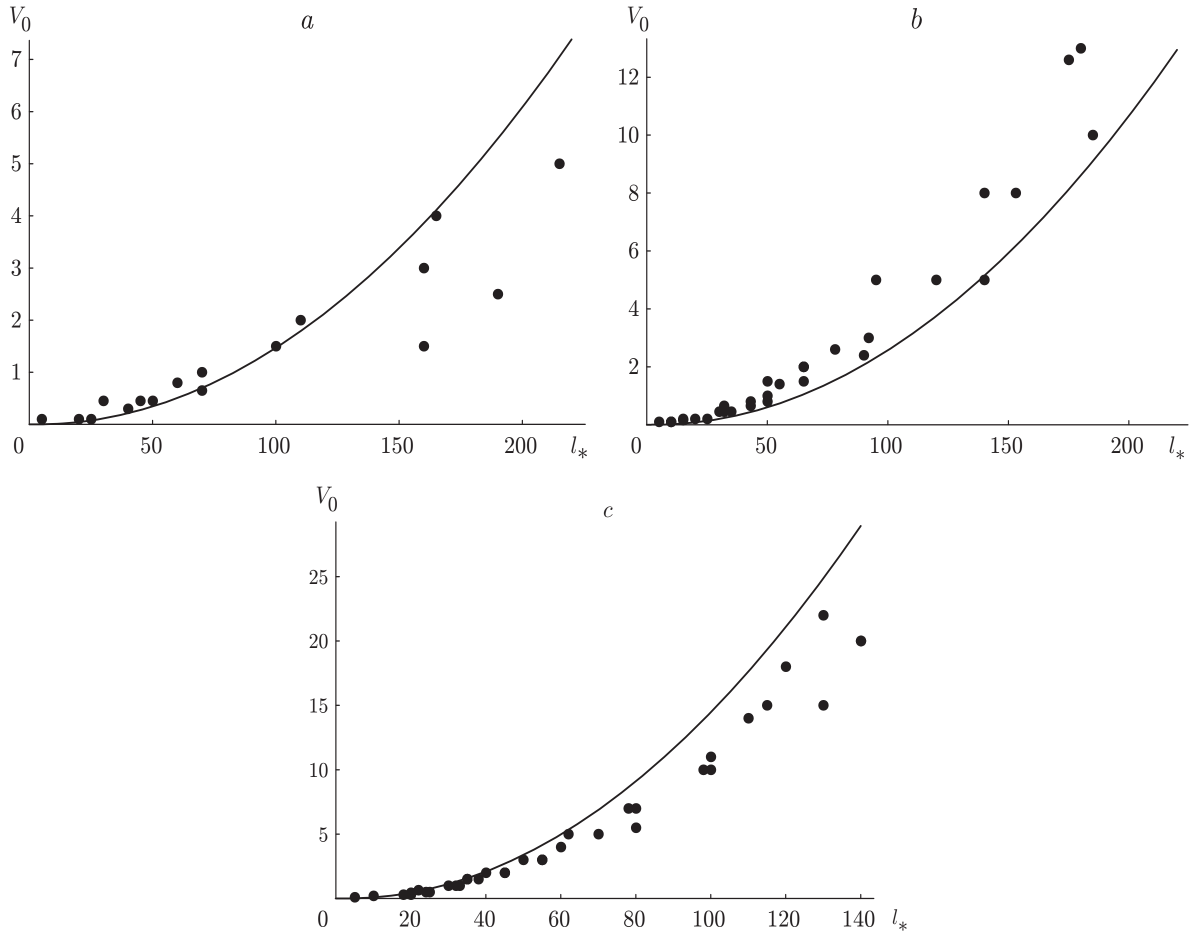

Fig. 3

Figure 2a–c shows the theoretical and experimental dependences $Q_0(l_*)$ for aluminum, glass-fiber plastic, and Plexiglas, respectively. The solid curves refer to formula (24) and the points are experimental data. One can see that the theoretical and experimental results are in good qualitative agreement; for large values of l_*, the results also agree quantitatively.

In the case of "stiff" loading, we consider the dependence of the deflection of the specimen end V_0 (see Fig. 1) on the length of the exfoliation zone l_*. We write formula (23) as

$$V_0 = \frac{Q_*}{bE}\,\frac{A_3 l_*^3 + A_2 l_*^2 + A_1 l_* + a_0}{1 + \gamma l_*},$$

$$A_1 = \frac{a_1}{2h}, \qquad A_2 = \frac{a_2}{(2h)^2}, \qquad A_3 = \frac{4}{(2h)^3}. \tag{26}$$

Figure 3a–c shows the theoretical and experimental dependences $V_0(l_*)$ for aluminum, glass-fiber plastic, and Plexiglas, respectively. The solid curves refer to formula (26) and the points are experimental data. The value of Q_* was set identical to the case of "soft" loading. The dependences are in good qualitative agreement. Quantitative agreement between theoretical and experimental data is observed for small deflections.

A comparison of theoretical and experimental results shows that the mathematical model proposed adequately describes the detachment of a beam glued to a rigid plate.

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